SLIIT

2023 SICASH
SLIIT INTERNATIONAL CONFERENCE ON ADVANCEMENTS IN SCIENCES & HUMANITIES

On Ptolemy's Theorem and Related Derivatives

[1]Indika Shameera Amarasinghe

[1]Department of Mathematics and Statistics, Faculty of Humanities and Sciences, SLIIT, Malabe, Sri Lanka

Corresponding author - *indika.a@sliit.lk

ARTICLE INFO

Article History:
Received: 10 September 2023
Accepted: 01 November 2023

Keywords:

Cyclic quadrilaterals; Equilateral triangles; Mathematical logic; Perpendiculars; Similar triangles.

Citation:

Indika Shameera Amarasinghe. (2023). On Ptolemy's Theorem and Related Derivatives. Proceedings of SLIIT International Conference on Advancements in Sciences and Humanities, 1-2 December, Colombo, pages 302-308.

ABSTRACT

In this paper an Euclidean Geometric proof is presented for the Ptolemy's Theorem of cyclic quadrilaterals by using a generalized identity with respect to a cevian of a triangle. Furthermore, a proof for the converse of the Ptolemy's Theorem is also presented, while adducing some significant applications, new corollaries and lemmas of Ptolemy's Theorem and its converse.

INTRODUCTION

The Ptolemy's Theorem of Cyclic Quadrilaterals founded and proved by Claudius Ptolemaeus who was an eminent Greek Mathematician, has been one of the prominent and exciting results in a geometry of a circle, throughout way back centuries ago, even at present not only in Advanced Geometry, but also in the other related sciences. There have been several alternative proofs for the Ptolemy's Theorem of cyclic quadrilaterals in the mathematics literature, using some geometric, trigonometric and non-geometric (Complex Number Algebra, Vector Algebra) approaches. The author himself has published a concise elementary proof for the Ptolemy's Theorem using only the Euclidean Geometry (without using trigonometry), proving some other useful properties in a cyclic quadrilateral in [1] in the references. In this paper, the

indika.a@sliit.lk

https://doi.org/10.54389/HBZJ1870

author adduces an alternative proof for the Ptolemy's Theorem of cyclic

quadrilaterals, involving a generalized corollary proved with respect to a cevian of a triangle, as well as for the converse of the Ptolemy's Theorem involving Mathematical Logic.

1. MATERIALS AND METHODS

Corollary 1

Let $ABC\Delta$ be an arbitrary plane triangle such that D be an arbitrary point on BC, with $BC = a$, $AC = b$ and $AB = c$. If AD is a cevian such that $\frac{BD}{DC} = \frac{1}{k}$ for some $k > 0$, then

$$AD^2 = \frac{(1+k)(b^2 + kc^2) - a^2k}{(k+1)^2}.$$

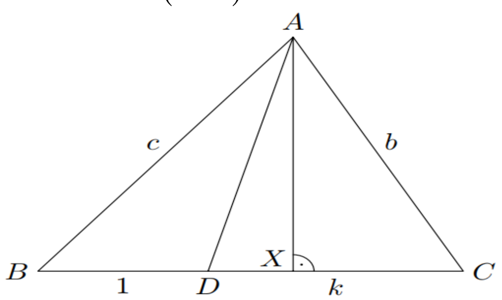


Figure 1. An Euclidean Triangle

Proof of corollary: The proof of the corollary is a conditional proof under proof by cases. For the sake of simplicity (or without loss of generality), assume that $ABC\Delta$ is an acute angle triangle.

Case 1. Assume that AD is not perpendicular to BC.

Proof:

Assume that AD is cevian such that $\frac{BD}{DC} = \frac{1}{k}$. Then draw the perpendicular AX to BC. Thus $DX \neq 0$. Using the Pythagoras Theorem respectively for $ABD\Delta$ (Obtuse Triangle), and $ADC\Delta$ (Acute Triangle), it follows that

$$c^2 = AD^2 - DX^2 + (BD + DX)^2 = AD^2 + BD^2 + 2BD.DX$$

, and

$$b^2 = AD^2 - DX^2 + (DC - DX)^2 = AD^2 + DC^2 - 2DC.DX.$$

These results lead us to $\frac{BD}{DC} = \frac{1}{k} = \frac{c^2 - AD^2 - BD^2}{AD^2 + DC^2 - b^2}$ since $k > 0$ and $DX \neq 0$.

Also, it is trivial to see that $BD = \frac{a}{k+1}$ and $DC = \frac{ka}{k+1}$. Thus, it follows that

$$\frac{1}{k} = \frac{c^2 - AD^2 - \left(\frac{a}{k+1}\right)^2}{AD^2 + \left(\frac{ka}{k+1}\right)^2 - b^2}$$

and after some elementary algebraic manipulation, this leads us to the desired result

$$AD^2 = \frac{(1+k)(b^2 + kc^2) - a^2k}{(k+1)^2}.$$

Case 2. Assume that AD is perpendicular to BC. (Now X is coincided with D)

Proof:

Then similarly, as before, using the Pythagoras Theorem, it follows $c^2 = a^2 + b^2 - 2a.DC$, as well as $b^2 = a^2 + c^2 - 2a.BD$.

Thus, it leads to $\frac{BD}{DC} = \frac{1}{k} = \frac{a^2 + c^2 - b^2}{a^2 + b^2 - c^2}$. Thus $k = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$.

Therefore

$$k+1=\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)+1=\frac{2a^2}{a^2+c^2-b^2}$$. Also, it follows $BD=\dfrac{a^2+c^2-b^2}{2a}$.

Then observe that

$$AD^2=c^2-BD^2=c^2-\left(\frac{a^2+c^2-b^2}{2a}\right)^2=\frac{4a^2c^2-\left(a^2+c^2-b^2\right)^2}{4a^2}.$$

Observe that

$$\frac{(1+k)\left(b^2+kc^2\right)-a^2k}{(k+1)^2}=\frac{\left(\frac{2a^2}{a^2+c^2-b^2}\right)\left(b^2+\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)c^2\right)-a^2\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)}{\left(\frac{2a^2}{a^2+c^2-b^2}\right)^2}==\frac{4a^2c^2-\left(a^2+c^2-b^2\right)^2}{4a^2}$$

$$=AD^2.$$

Hence it follows that in each case,

$$AD^2=\frac{(1+k)\left(b^2+kc^2\right)-a^2k}{(k+1)^2}.$$ Now it is not difficult to deduce that, if $ABC\Delta$ is an obtuse triangle then also $AD^2=\dfrac{(1+k)\left(b^2+kc^2\right)-a^2k}{(k+1)^2}$.

2. RESULTS AND DISCUSSIONS

Theorem 1 (Ptolemy's Theorem)

If $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals then $AC \cdot BD = AB \cdot DC + AD \cdot BC$. This is referred to as the Ptolemy's Theorem of Cyclic Quadrilaterals.

Proof *(New Proof)*. Assume that $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals. Suppose $AB=a$ $BC=b$, $CD=c$ and $AD=d$. Let E be the point of intersection of the diagonals AC and BD, and let $\dfrac{BE}{ED}=\dfrac{1}{k}$ and $\dfrac{AE}{EC}=\dfrac{1}{m}$ for some constants $k, m>0$.

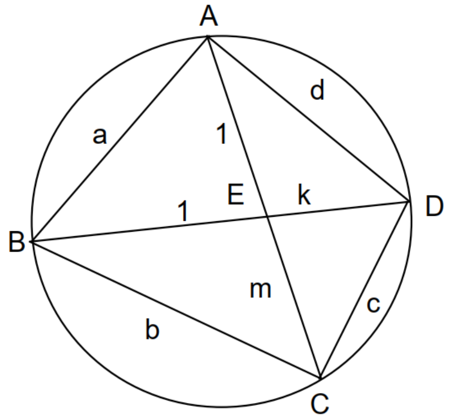


Figure 2. A Cyclic Quadrilateral

Since , and , $ABE\Delta$ and $EDC\Delta$ are similar. Hence $\dfrac{BE}{EC}=\dfrac{a}{c}$.

Since , and , $AED\Delta$ and $BEC\Delta$ are similar. Hence $\dfrac{AE}{BE}=\dfrac{ED}{EC}=\dfrac{d}{b}$. Thus $\left(\dfrac{BE}{EC}\right)\left(\dfrac{AE}{BE}\right)=\left(\dfrac{a}{c}\right)\left(\dfrac{d}{b}\right)$ which leads to $\dfrac{AE}{EC}=\dfrac{ad}{bc}=\dfrac{1}{m}$. Hence $m=\dfrac{bc}{ad}$.

Also, observe that

$$\frac{\left(\frac{BE}{EC}\right)}{\left(\frac{ED}{EC}\right)}=\frac{\left(\frac{a}{c}\right)}{\left(\frac{d}{b}\right)}.$$ Therefore, $\dfrac{BE}{ED}=\dfrac{ab}{cd}=\dfrac{1}{k}$. Hence $k=\dfrac{cd}{ab}$. Then by using the above corollary on cevians to $ABD\Delta$, we yield

$$AE^2=\frac{(1+k)\left(d^2+ka^2\right)-BD^2k}{(k+1)^2}.$$ Similarly, by using the above corollary to $BCD\Delta$

, we yield $EC^2 = \frac{(1+k)(c^2+kb^2) - BD^2k}{(k+1)^2}$. These two results lead to $\frac{AE^2}{EC^2} = \frac{(1+k)(d^2+ka^2) - BD^2k}{(1+k)(c^2+kb^2) - BD^2k} = \frac{1}{m^2}$. By simplifying, this leads to $BD^2k(m^2-1) = (k+1)(m^2d^2 + m^2a^2k - c^2 - kb^2)$. By substituting the above values for k and m, this leads to

$$BD^2\left(\frac{cd}{ab}\right)\left(\left(\frac{bc}{ad}\right)^2 - 1\right) = \left(\left(\frac{cd}{ab}\right)+1\right)\left(\left(\frac{bc}{ad}\right)^2 d^2 + \left(\frac{bc}{ad}\right)^2 a^2\left(\frac{cd}{ab}\right) - c^2 - \left(\frac{cd}{ab}\right)b^2\right).$$

By simplifying we have $BD^2(bc-ad)(bc+ad) = (ab+cd)(bd+ac)(bc-ad)$.

Case 1. Now assume that $bc \neq ad$.

Then it easily follows $BD^2 = \frac{(ab+cd)(ac+bd)}{(ad+bc)}$.

It is trivial to see that $AE = \frac{AC}{m+1}$ and $EC = \frac{mAC}{m+1}$. Then observe that

$$AE^2 - EC^2 = \frac{(1+k)(d^2+ka^2) - BD^2k}{(k+1)^2} - \left[\frac{(1+k)(c^2+kb^2) - BD^2k}{(k+1)^2}\right] = \frac{k(a^2-b^2)+d^2-c^2}{k+1} =$$

$$\left(\frac{AC}{m+1}\right)^2 - \left(\frac{mAC}{m+1}\right)^2 = \frac{k(a^2-b^2)+d^2-c^2}{k+1} = AC^2\left(\frac{1-m^2}{(m+1)^2}\right) = AC^2\left(\frac{1-m}{1+m}\right).$$

By substituting the above values for k and m, this leads to

$$AC^2\left(\frac{1-\left(\frac{bc}{ad}\right)}{\left(1+\frac{bc}{ad}\right)}\right) = \frac{\left(\frac{cd}{ab}\right)(a^2-b^2)+d^2-c^2}{\left(\frac{cd}{ab}\right)+1}$$

. Hence $AC^2\frac{(ad-bc)}{ad+bc} = \frac{(ac+bd)(ad-bc)}{ab+cd}$. Since by our assumption, $bc \neq ad$, it easily follows that $AC^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}$.

Case 2. Now assume that $bc = ad$.

Then since $m = \frac{bc}{ad}$, it follows $m = 1$. That is, then E is the midpoint of AC.

Then by using the Apollonius Theorem for the $ADC\Delta$, it follows that $2AE^2 + 2ED^2 = d^2 + c^2$.

Observe that by the above-mentioned similar triangles $ED = EC\left(\frac{d}{b}\right)$, and $EC = \frac{mAC}{m+1}$, it follows that

$$ED = \left(\frac{mAC}{m+1}\right)\left(\frac{d}{b}\right) = \left(\frac{\left(\frac{bc}{ad}\right)AC}{\left(\frac{bc}{ad}\right)+1}\right)\left(\frac{d}{b}\right) = \frac{AC\cdot cd}{bc+ad}$$

. Moreover, $AE = \frac{AC}{m+1} = \frac{AC}{\left(\frac{bc}{ad}\right)+1} = \frac{ACad}{ad+bc}$.

Thus, by the above Apollonius Theorem, it follows that $2\left(\frac{ACad}{ad+bc}\right)^2 + 2\left(\frac{AC\cdot cd}{bc+ad}\right)^2 = d^2 + c^2$. By simplifying this further, since $bc = ad$, and rearranging the terms, we yield to the desired result

$$AC^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}.$$

Observe that $EC = BE\left(\frac{c}{a}\right)$. Since $BE = \frac{BD}{k+1}$, it follows $EC = \left(\frac{BD}{k+1}\right)\left(\frac{c}{a}\right)$. Then from the above proved relation, we have

$$EC^2 = \frac{(1+k)(c^2+kb^2) - BD^2 k}{(k+1)^2} = \left(\frac{BDc}{a(k+1)}\right)^2$$

. Substituting for k, we have

$$\frac{BD^2c^2}{a^2\left(\frac{cd}{ab}+1\right)^2} = \frac{\left(1+\frac{cd}{ab}\right)\left(c^2+\left(\frac{cd}{ab}\right)b^2\right) - BD^2\left(\frac{cd}{ab}\right)}{\left(\frac{cd}{ab}+1\right)^2}$$

which leads to $BD^2 = \frac{(ab+cd)(ac+bd)}{(ad+bc)}$.

That is in each case $AC^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}$ and $BD^2 = \frac{(ab+cd)(ac+bd)}{(ad+bc)}$. Hence, we yield

$$AC^2 \cdot BD^2 = \frac{(ad+bc)(ac+bd)}{ab+cd} \times \frac{(ab+cd)(ac+bd)}{(ad+bc)} = (ac+bd)^2$$

.

Hence it easily follows $AC \cdot BD = AB \cdot DC + AD \cdot BC$ which is the Ptolemy's Theorem of Cyclic Quadrilaterals. This completes the proposed alternative proof of Ptolemy's Theorem.

Remark 1. It also follows that $\frac{AC}{BD} = \frac{ad+bc}{ab+cd}$.

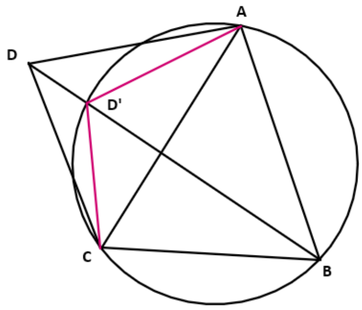
Lemma 1. Assume that $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals, and $AB = a$ $BC = b$, $CD = c$ and $AD = d$. Then the intersection point E of the diagonals is the midpoint of AC **if and only if** $bc = ad$.

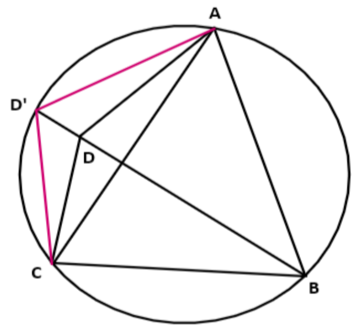
Proof of Lemma 1. Proof is trivial under the above case 2, if $m = 1$.

The Converse of the Ptolemy's Theorem

Let A, B, C and D be four arbitrary points in a plane. If $AC \cdot BD = AB \cdot DC + AD \cdot BC$ such that AC and BD are the diagonals of the quadrilateral $ABCD$, then the points A, B, C and D are on a circle.

Proof. Proof is a proof by ***contraposition*** & **proof by cases**. Assume that **at least one** point of A, B, C and D is not on a circle. Without loss of generality, assume that D is not on the circle.





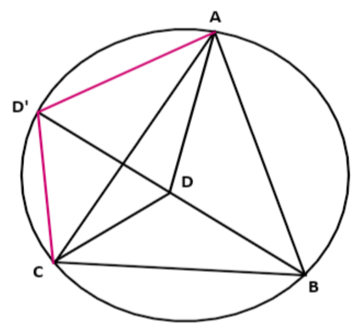


Figure 3. D is outside circle Figure 4. D

is inside circle Figure 5. D is inside $ABC\Delta$

Case 1. Assume that D is outside the $ABC\Delta$ and the circumcircle of $ABC\Delta$ (figure 3).

Proof. Using the Ptolemy's Theorem, it follows $AC \cdot BD' = AB \cdot D'C + AD' \cdot BC$. Since is an obtuse angle, by the very elementary geometry, it is trivial to see that . Thus $AD > AD'$. Similarly, it follows $CD > CD'$. Also, $BD > BD'$. Therefore by writing $BD' = BD - DD'$, due to the **arbitrariness** of $DD' > \dot{0}$, it follows that $AC \cdot BD < AB \cdot DC + AD \cdot BC$, that is $AC \cdot BD \neq AB \cdot DC + AD \cdot BC$. Thus, by **contraposition**, the converse of the Ptolemy's Theorem is proved.

Case 2. Assume that D is outside the $ABC\Delta$, but is inside the circumcircle of $ABC\Delta$. (See figure 4)

Proof. Using the Ptolemy's Theorem, it follows $AC \cdot BD' = AB \cdot D'C + AD' \cdot BC$. Similarly, as in case 1, by using the very elementary geometry, it follows that $AD < AD'$, $CD < CD'$ and $BD < BD'$. In addition, $BD' = BD + DD'$. Due to the **arbitrariness** of $DD' > \dot{0}$, this leads us to $AC \cdot BD < AB \cdot DC + AD \cdot BC$, that is, $AC \cdot BD \neq AB \cdot DC + AD \cdot BC$. Thus, by **contraposition**, the converse of the Ptolemy's Theorem is proved.

Case 3. Assume that D is inside the $ABC\Delta$, and inside the circumcircle of $ABC\Delta$. (See figure 5)

Proof. Using the Ptolemy's Theorem, it follows $AC \cdot BD' = AB \cdot D'C + AD' \cdot BC$. In this case it is possible that $AD = AD'$ and $CD = CD'$, OR $AD < AD'$ and $CD < CD'$, OR $AD > AD'$ and $CD > CD'$. But since $BD < BD'$, even if $AD = AD'$ and $CD = CD'$, it follows that $AC \cdot BD < AB \cdot DC + AD \cdot BC$. In the rest of the cases, $AD \neq AD'$ and $CD \neq CD'$, similarly, as in the above case 2 and case 1, it follows that $AC \cdot BD \neq AB \cdot DC + AD \cdot BC$. Thus, in **all** possible cases, it follows that $AC \cdot BD \neq AB \cdot DC + AD \cdot BC$. Thus, by **contraposition**, the converse of the Ptolemy's Theorem is proved.

Lemma 2. (Under the converse of Ptolemy's Theorem)

Let $ABC\Delta$ is an equilateral triangle in a plane. Assume that the point D is outside the $ABC\Delta$ being on the same plane such that AC and BD are the diagonals of the quadrilateral $ABCD$ with $BD = AD + DC$. Then the points A, B, C and D are on a circle.

Proof. Since $ABC\Delta$ is an equilateral triangle, it follows that $AB = BC = AC$. Also, since it is given that $BD = AD + DC$, it follows $AC \cdot BD = BC \cdot AD + AB \cdot DC$. Hence, by the converse of the Ptolemy's Theorem, it follows that the points A, B, C and D are on a circle.

Remark 2. Observe that the converse of the above Lemma 2 is a very well-established result in circle geometry.

3. CONCLUSIONS

In this paper, the Ptolemy's Theorem of Cyclic Quadrilaterals is proved by a different approach using a derived identity around a cevian of a triangle. One may feel that since the author himself has already given a shorter proof of the same theorem in the literature (in [1]), it is redundant to present another proof of it using a lengthier approach rather than his previous proof. But the readers are encouraged to analyse the author's novel approach of the proof of the Ptolemy's Theorem presented here, as it leads to many other significant and important new corollaries and lemmas in being attempted to prove the Ptolemy's Theorem in this way. Moreover, the converse of the Theorem is proved by using the contraposition and proof by cases is also important since it is hard to find complete proofs for the converse of the Ptolemy's Theorem in an Euclidean Geometric way.

REFERENCES

Amarasinghe, I. (2014). A Concise Elementary Proof for the Ptolemy's Theorem. *Global Journal of Advanced Research on Classical and Modern Geometries (GJARCMG)*, 20-25.

Amarasinghe, I. (2011). A New Theorem on any Right-angled Cevian Triangle. *Journal of the World Federation of National Mathematics Competitions (JWFNMC)*, 29-37.

Alsina, C & Nelson, R.B. (2007). On the Diagonals of a Cyclic Quadrilateral. *Forum Geometricorum*, 147-149.

Amarasinghe, I. (2010). New Proof for Stewart's Ptolemy and Apollonius Theorems by Various Point of Views. *African Journal of Mathematics and Computer Science Research* (Accepted).

Amarasinghe, I. (2012). On the Standard Lengths of Angle Bisectors and the Angle Bisector Theorem. *Global Journal of Advanced Research on Classical and Modern Geometries (GJARCMG)*, 15-27.

Shisha, O. (1991). On Ptolemy's Theorem. *International Journal of Mathematics and Mathematical Sciences*, 410.

Apostol, T. (1967). Ptolemy's Inequality and the Chordal Metric. *Mathematics Magazine*, 233 – 235.

Aliyev, S, Hamidova, S & Abdullayeva, G. (2020). Some Applications in Ptolemy's Theorem in Secondary School Mathematics. *European Journal of Pure and Applied Mathematics*, 180 – 184.