k – Graceful Labeling of Triangular Type Grid Graphs $D_n(P_m)$ and L – Vertex Union of $D_n(P_m)$

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ABSTRACT

Graph labeling is one of the most popular research topics in the field of graph theory. Prime labeling, antimagic labeling, radio labeling, graceful labeling, lucky labeling, and incidence labeling are some of the labeling techniques. Among the above-mentioned techniques, graceful labeling is one of the most engaging graph labeling techniques with a vast amount of real-world applications. Over the past few decades, plenty of studies have been conducted on this area in various dimensions. Grid graphs are very much useful in applications of circuit theory, communication networks, and transportation networks. However, in the literature, there are not many research papers on the graceful labeling of grid graphs except a few on odd graceful labeling. In our work, we prove that triangular-type grid graphs, $D_n(P_m)$ and L – vertex union of $D_n(P_m)$ admit k – general graceful labeling and k – even and k – odd graceful labeling. Further, we introduce combinatorial proofs for them as well.

KEYWORDS: $\boldsymbol{k}$ – *even graceful labeling,* $\boldsymbol{k}$ – *graceful labeling,* $\boldsymbol{k}$ – *odd graceful labeling, triangular type grid graph.*

1 INTRODUCTION

The graceful labeling method was put forward by Alexander Rosa in 1967 this labeling method was originally given the name as β – labeling and later it was named graceful labeling by Golomb (Gallian, 2009). There is a very famous conjecture called the graceful tree conjecture abbreviated as GTC which hypothesizes that all trees are graceful and it remains unsolved. However, the notion of graceful labeling was very much used for decomposing the complete graphs into isomorphic subgraphs. There are a lot of real-world applications of graceful labeling in coding theory, missile guiding codes, x-ray crystallography, cryptography, etc (Deshmukh, 2015).

In our work, we use the concept of natural generalization of graceful labeling which is known as k – graceful labeling which was introduced around 1982.

Despite the large number of publications carried out on the title of graceful labeling, research paper related to the graceful labeling of grid graphs is a bit rare in the literature. In 2013, M. E. Abdel-Aal discusses the odd graceful labeling of $D_n(P_m)$ in a paper on new classes of odd graceful graphs (Abdel-Aal, 2013). S.K. Vaidya and B. Lekha published an article on new families of odd graceful graphs, and there they discuss odd graceful labeling of $D_2(P_m)$ around 2010 (Vaidya & Lekha, 2010). So, we are interested to introduce k – graceful labeling and k – even and k – odd graceful labeling of $D_n(P_m)$ and L – vertex union of $D_n(P_m)$ hoping to apply these labelings to real-world problems in future work. In cryptography for encrypting messages and communication networks for multi-protocol label switching networks.

2 METHODOLOGY

First, we begin by giving the notations and the definitions used in this research work.

Definition 1. k – Graceful labeling. Let $G = G(V, E)$ be a graph with $p = |E(G)|$, where $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Graceful labeling of G is a vertex labeling $f: V \to [0, p + k - 1]$ such that f is injective and the edge labeling $f_\gamma: E \to [1, p + k - 1]$ defined by

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$f_\gamma(uv) = |f(u) - f(v)|$ is also injective. Such a graph is called a k – graceful graph. When $k = 1$, it is called general graceful labeling.

Definition 2. k – even graceful labeling. A graph $G = (V, E)$ with q edges is said to admit even graceful labeling if there is an injection, $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q\}$ such that when each xy edge is assigned the label $|f(x) - f(y)|$. A graph that admits even graceful labeling is called an even graceful graph.

Definition 3. k – odd graceful labeling. A graph $G = (V, E)$ with q edges is said to admit odd graceful labeling if there is an injection, $f : V(G) \rightarrow \{0, 1, 2, \ldots, (2q - 1)\}$ such that when each xy edge is assigned the label $|f(x) - f(y)|$. A graph that admits odd graceful labeling is called an odd graceful graph.

Definition 4. The triangular-type grid graph $D_n(\mathrm{G})$ of a connected graph G is constructed by taking n copies of G, say $G_1, G_2, G_3, G_4, \ldots, G_n$, then join each vertex $U_{i,j}$ with $U_{i\pm1,j+p}$ where $1 - j \leq p \leq n - j$ only in G_i where each G_i is a P_m, path graph of m vertices such that $1 \leq i \leq m$ and $1 \leq j \leq n$, more precisely $D_n(P_m)$. Let $D_n(P_m)$ be a graph with $mn = |V(G)|$, $n^2(\mathrm{m} - 1) = |E(G)|$ where $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively.

Theorem 1: The graph $D_n(P_m)$ is k – graceful for any finite m and n.

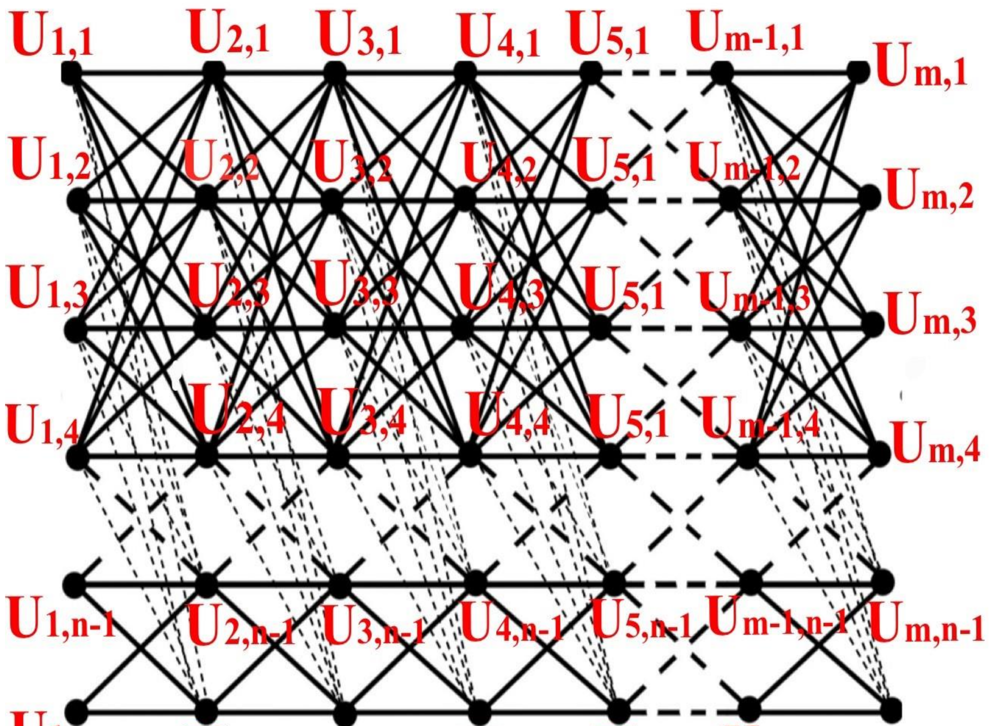

Figure 1. The graph $D_n(P_m)$

Let f be the vertex k – graceful labeling of the above grid graph,

$$f(U_{i,j}) = \begin{cases} \frac{i-1}{2} n^2 + j - 1; \;\; i - odd; \;\; j - 1,2,\ldots,n \\ n^2(m-1) + n^2 \frac{i-2}{2} - n(j-1) + k - 1; \; i - even, j - 1,2,\ldots,n, k \in \mathbb{N} \end{cases} \quad (1)$$

Now, we introduce the formula obtained for the k – even graceful labeling for the $D_n(P_m)$.

Theorem 2: The graph $D_n(P_m)$ is k – even graceful for any finite m and n.

Let f be the vertex k – even graceful labeling of the above grid graph,

$$f(U_{i,j}) = \begin{cases} (i-1)n^2 + 2(j-1); \;\; i - odd, \; j - 1,2,\ldots,n \\ n^2\{2(m-1) + i - 2\} - 2n(j-1) + k - 2; \;\; i, k - even, j - 1,2,\ldots,n \end{cases} \tag{2}$$

Next, we introduce an alternative formula for obtaining k - odd graceful labeling for the $D_n(P_m)$.

Theorem 3: The graph $D_n(P_m)$ is k – odd graceful for any finite m and n.

Let f be the vertex k – odd graceful labeling of the above grid graph,

$$f(U_{i,j}) = \begin{cases} (i-1)n^2 + 2(j-1); \;\; i - odd, \; j - 1,2,\ldots,n \\ n^2\{2(m-1) + i - 2\} - 2n(j-1) + k - 2; \; i - even, k - odd, j - 1,2,\ldots,n \end{cases} \tag{3}$$

For further investigations, we consider a diagonal extended version of $D_n(P_m)$, we call it L – vertex union of $D_n(P_m)$.

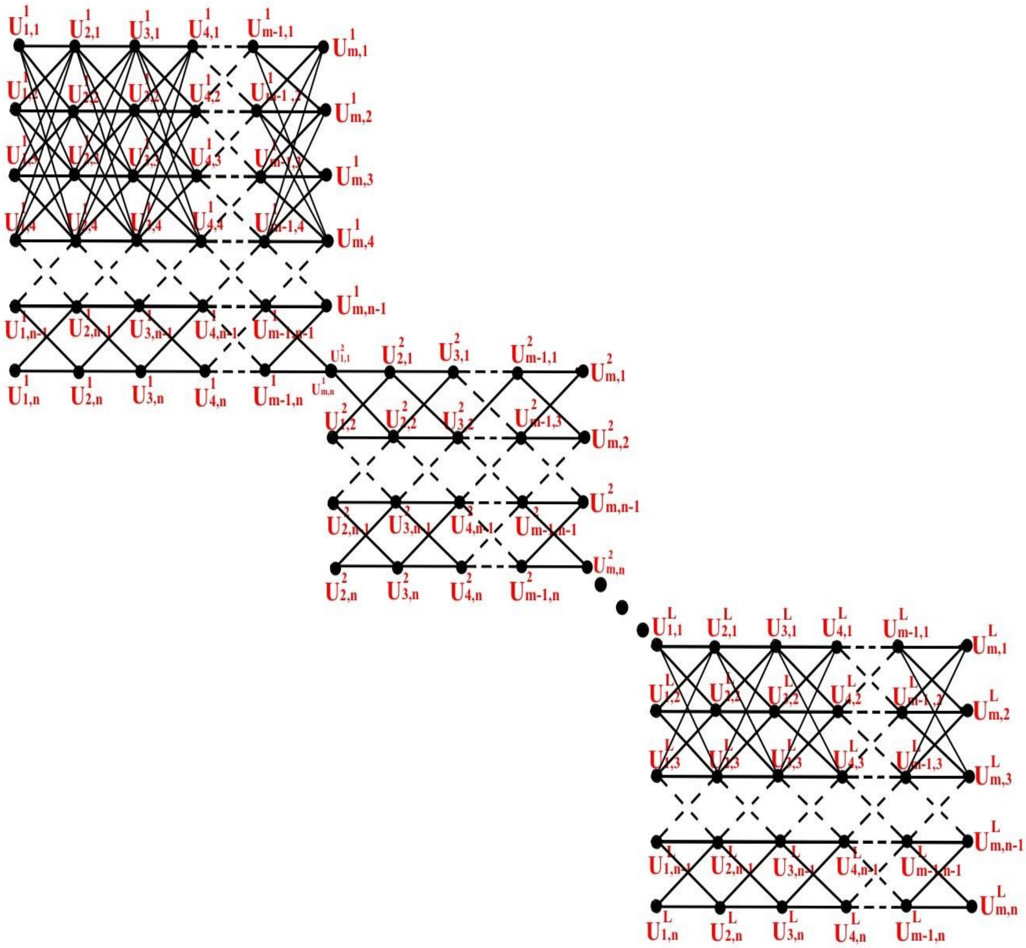

Figure 2. L – vertex union of $D_n(P_m)$

Theorem 4: L – vertex union of $D_n(P_m)$ is k – graceful for any finite m and n.

Let f be the vertex k – graceful labeling of the above grid graph. The formula of the k – graceful labeling for the first$(L=1)$ grid,

$$f\left(U_{i,j}^{(L=1)}\right)=\begin{cases}\frac{i-1}{2}n_L^2+j-1;\ i-odd, j-1,2,\ldots,n_L\\ \sum_{L=1}^{l} n_L^2(m_L-1)+\frac{n_L^2(2-i)}{2}-n_L(j-1)+k-1;\ i-even, j-1,2,\ldots,n_L\end{cases} \quad (4)$$

Note that, when all the grids are the same in size, $\sum_{L=1}^{l} n_L^2(m_L-1)=n^2(m-1)l$, where l is the total number of grids.
The formula can be divided into two parts depending on the following factor of the k – graceful labeling for the $L>1$ grids,

Case 1: $U_{m_{L-1},n_{L-1}}^{L-1}-U_{m_{L-1},n_{L-1}-1}^{L-1}=1$

$$f\left(U_{i,j}^{L}\right)=\begin{cases}U_{m_{L-1},n_{L-1}}^{L-1}+\frac{i-1}{2}n_L^2+j-1;\ i-odd, j-1,2,\ldots,n_L\\ f\left(U_{m_{L-1}-1,n_{L-1}}^{L-1}\right)+\frac{U_{m_{L-1},n_{L-1}}^{L-1}-U_{m_{L-1},n_{L-1}-1}^{L-1}}{\left|U_{m_{L-1}-1,n_{L-1}}^{L-1}-U_{m_{L-1}-1,n_{L-1}-1}^{L-1}\right|}\left\{1+\frac{n_L^2(i-2)}{2}+n_L(j-1)\right\};\ i-even, j-1,2,\ldots,n_L\end{cases}$$
(5)

Case 2: $U_{m_{L-1},n_{L-1}}^{L-1}-U_{m_{L-1},n_{L-1}-1}^{L-1}\neq 1$

$$f\left(U_{i,j}^{L}\right)=\begin{cases}f\left(U_{m_{L-1},n_{L-1}}^{L-1}\right)+\frac{U_{m_{L-1},n_{L-1}}^{L-1}-U_{m_{L-1},n_{L-1}-1}^{L-1}}{\left|U_{m_{L-1},n_{L-1}}^{L-1}-U_{m_{L-1},n_{L-1}-1}^{L-1}\right|}\left\{\frac{i-1}{2}n_L^2+n_L(j-1)\right\};\ i-odd, j-1,2,\ldots,n_L\\ f\left(U_{m_{L-1}-1,n_{L-1}}^{L-1}\right)+\frac{n_L^2(i-2)+2j}{2};\ i-even,\ j-1,2,\ldots,n_L\end{cases}$$
(6)

Theorem 5: L – vertex union of $D_n(P_m)$ is k – odd graceful for any finite m and n.

Let f be the vertex k – odd graceful labeling of the above grid graph. The formula of the k – odd graceful labeling for the first$(L=1)$ grid,

$$f\left(U_{i,j}^{(L=1)}\right)=\begin{cases}(i-1)n_L^2+2(j-1);\ i-odd, j-1,2,\ldots,n_L\\ 2\sum_{L=1}^{l} n_L^2(m_L-1)+n_L^2(2-i)-2n_L(j-1)+k-2;\ i-even, j-1,2,\ldots,n_L, k-odd\end{cases} \quad (7)$$

Theorem 6: L – vertex union of $D_n(P_m)$ is k – even graceful for any finite m and n.

Let f be the vertex k – even graceful labeling of the above grid graph. The formula of the k – even graceful labeling for the first$(L=1)$ grid,

$$f\left(U_{i,j}^{(L=1)}\right)=\begin{cases}(i-1)n_L^2+2(j-1);\ \ i-odd, j-1,2,\ldots,n_L\\ 2\sum_{L=1}^{l} n_L^2(m_L-1)+n_L^2(2-i)-2n_L(j-1)+k-2;\ \ i,k-even, j-1,2,\ldots,n_L\end{cases}$$
(8)

The formula for k – odd and even graceful labeling of the $D_n(P_m)$ for the $L>1$ grid are the same and can be divided into two parts depending on the following factor of the k – graceful labeling for the $L>1$ grid,

Case 1: $U_{m_{L-1},n_{L-1}}^{L-1}-U_{m_{L-1},n_{L-1}-1}^{L-1}=2$

$$f\left(U_{i,j}^{L}\right) = \begin{cases} U_{m_{L-1},n_{L-1}}^{L-1} + (i-1)n_L^2 + 2(j-1); \; i-odd, j-1,2,\ldots,n_L \\ f\left(U_{m_{L-1}-1,n_{L-1}}^{L-1}\right) + \frac{f\left(U_{m_{L-1}-1,n_{L-1}}^{L-1}\right)-f\left(U_{m_{L-1}-1,n_{L-1}-1}^{L-1}\right)}{\left|f\left(U_{m_{L-1}-1,n_{L-1}}^{L-1}\right)-f\left(U_{m_{L-1}-1,n_{L-1}-1}^{L-1}\right)\right|}\{2(1+n_L(j-1))+n_L^2(i-2)\}; i-even, j-1-n_L \end{cases} \quad (9)$$

Case 2: $U_{m_{L-1},n_{L-1}}^{L-1} - U_{m_{L-1},n_{L-1}-1}^{L-1} \neq 2$

$$f\left(U_{i,j}^{L}\right) = \begin{cases} f\left(U_{m_{L-1},n_{L-1}}^{L-1}\right) + \frac{f\left(U_{m_{L-1},n_{L-1}}^{L-1}\right)-f\left(U_{m_{L-1},n_{L-1}-1}^{L-1}\right)}{\left|f\left(U_{m_{L-1},n_{L-1}}^{L-1}\right)-f\left(U_{m_{L-1},n_{L-1}-1}^{L-1}\right)\right|}\{(i-1)n_L^2 + 2n_L(j-1)\}, i-odd, j-1,2,\ldots,n_L \\ f\left(U_{m_{L-1}-1,n_{L-1}}^{L-1}\right) + \frac{n_L^2(i-2)+2j}{2}, i-even, \; j-1,2,\ldots,n_L \end{cases} \quad (10)$$

3 RESULTS AND DISCUSSION

The following figures illustrate the above-mentioned theorems and we use those figures to prove that triangular type grid graph and its diagonal extended version admit graceful labeling.

Proof of Theorem 1:

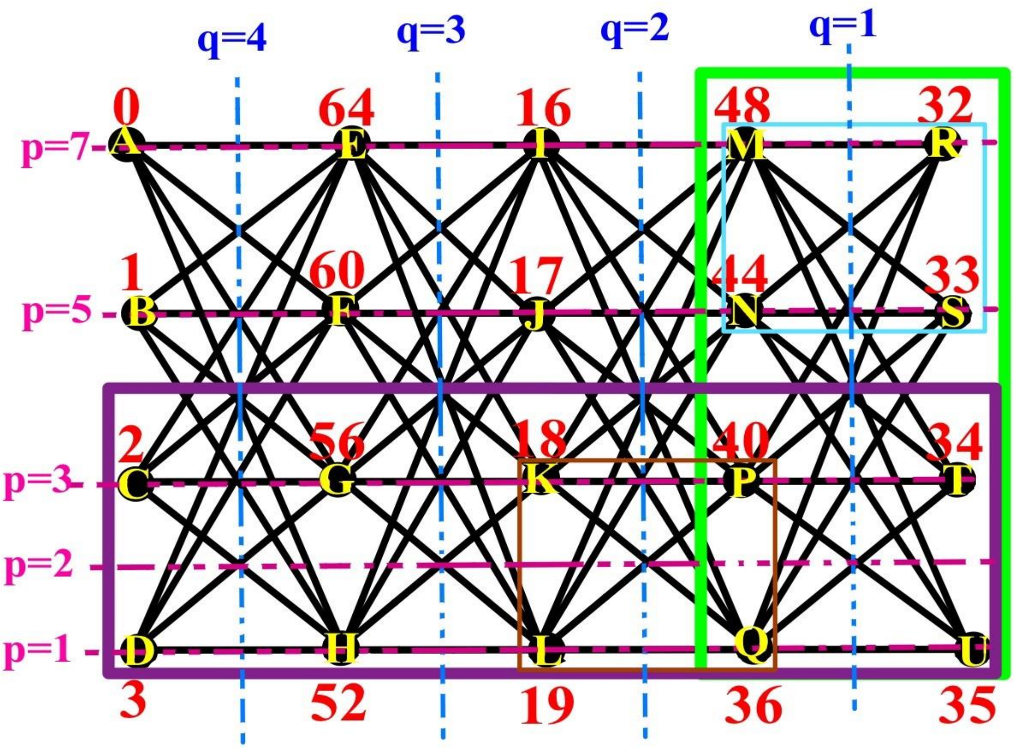

Figure 3. 1 - Graceful labeling of the $D_n(P_m)$

In order to show that graph $D_n(P_m)$ admits graceful labeling, we have to show that no edge label repeats, the maximum edge label that can have is the total number of edges of the graph, and each edge gets a label.

Here, we introduce new reference labels to edge labels for constructing formulas for the edge labels using p and q, and vertices are labeled using block letters for reference edges easily. To obtain the corresponding p or q value of a particular edge set, find the midpoints of that edge set and follow the dotted lines. All the following equations are obtained for considering a column.

First, consider the green column. Considering the horizontally parallel edge labels, QU, PT, NS, MR through $p = 1, 3, 5,$ and 7, we can introduce the following formula for all the horizontal parallel edges.

$$The\ edge\ labels\ =\ bottom\ edge\ label + (n+1)\frac{p-1}{2}, p = 1, 3, 5, 7, \ldots \quad (11)$$

Then, considering the zig-zag patterned edges like QT, TN, NR and UP, PS, SM through $p = 2, 4, 6$. We can introduce the following formula,

$$The\ edge\ labels\ =\ bottom\ edge\ label + (n+1)\frac{p-2}{2}, p = 2, 4, 6, \ldots \quad (12)$$

Finally, consider any set of slanted parallel edges like QS, PR, and UN, IM through $p = 3, 5$. We can introduce the following formula for those parallel edges.

$$The\ edge\ labels = bottom\ edge\ label + (n+1)\frac{p-3}{2},\ p = 3, 5, \ldots \quad (13)$$

Since all the Eqs. (11)-(13) are monotonically increasing with p, we can figure out that the highest edge labels must be at the top of any column(green) and minimum edge labels must be at the bottom of any column(green).

Now, consider the purple row. Considering the horizontal edge labels like QU, LQ, HL, DH and TP, PK, KG, GC and the zig-zag patterned edges like UP, PL, LG, GD and TQ, QK, KH, HC through $q = 1, 2, 3, 4$. We can introduce the following formula,

$$The\ edge\ labels = bottom\ edge\ label + n^2(q-1),\ q = 1, 2, 3, 4, \ldots \quad (14)$$

Since Eq. (14) is monotonically increasing with q, we can figure out that the highest edge labels must be on the left-most side of any row(purple) and minimum edge labels must be on the rightmost side of any row(purple).

All in all, we can say that the maximum edge label given by these formulas must be the left-most upper edge label, $U_{1,1}U_{2,1}$ according to Figure 1.

Consider the two squares light blue and brown. The light blue square is the top of the green column and it must contain the highest edge labels of the edges in that column can have. It contains edge labels $11, 12, 15, and\ 16$. $13, 14$ edge labels are in the diagonals of this entire column. Now, consider the brown square and it is the bottom of that corresponding column. So, it must contain the minimum edge labels of the edges in that column can have and it contains $22, 21, 18, and\ 17$. So, the brown square's element is greater than any element in the light blue square. Consequently, this condition is valid for edge labels in any two columns too. This implies that no edge label repeats.

Using the equation 1 for horizontal parallel edge labels through $p = 1, 3, 5, \ldots, 2n-1$, we can obtain the edge label of the $U_{m-1,1}U_{m,1}$ in Figure 1 as follows,

$$The\ edge\ label\ of\ U_{m-1,1}U_{m,1}\ =\ 1 + (n+1)\frac{p-1}{2} \quad (15)$$

Here $p = 2n - 1$. $\therefore\ The\ edge\ label\ of\ U_{m-1,1}U_{m,1}\ =\ 1 + (n+1)(n-1) = n^2$. (16)

Using the equation 4 for horizontal edge labels through $q = 1, 2, 3, \ldots, m-1$, we can obtain the edge label of the $U_{1,1}U_{2,1}$ in Figure 1 as follows,

$$The\ edge\ label\ of\ U_{1,1}U_{2,1} = n^2 + n^2(q-1) \quad (17)$$

Here $q = m - 1$. $\therefore The\ edge\ label\ of\ U_{1,1}U_{2,1} = n^2 + n^2(m-2) = n^2(m-1)$. (18)

Now, we can confirm that the maximum edge label given by these formulas does not exceed the total number of edges of this graph in Figure 1.

Since we label edges starting from 1 and up until the total number of the edges of that graph and have considered each and every edge label and confirmed that no edge label repeats and the maximum edge label does not exceed the total number of edges, we can conclude that all the edges of that graph must be labeled from 1 up to the total number of edges without repeating. Proof of theorems 2 and 3 can be proved by using the same procedure.

Proof of Theorem 4:

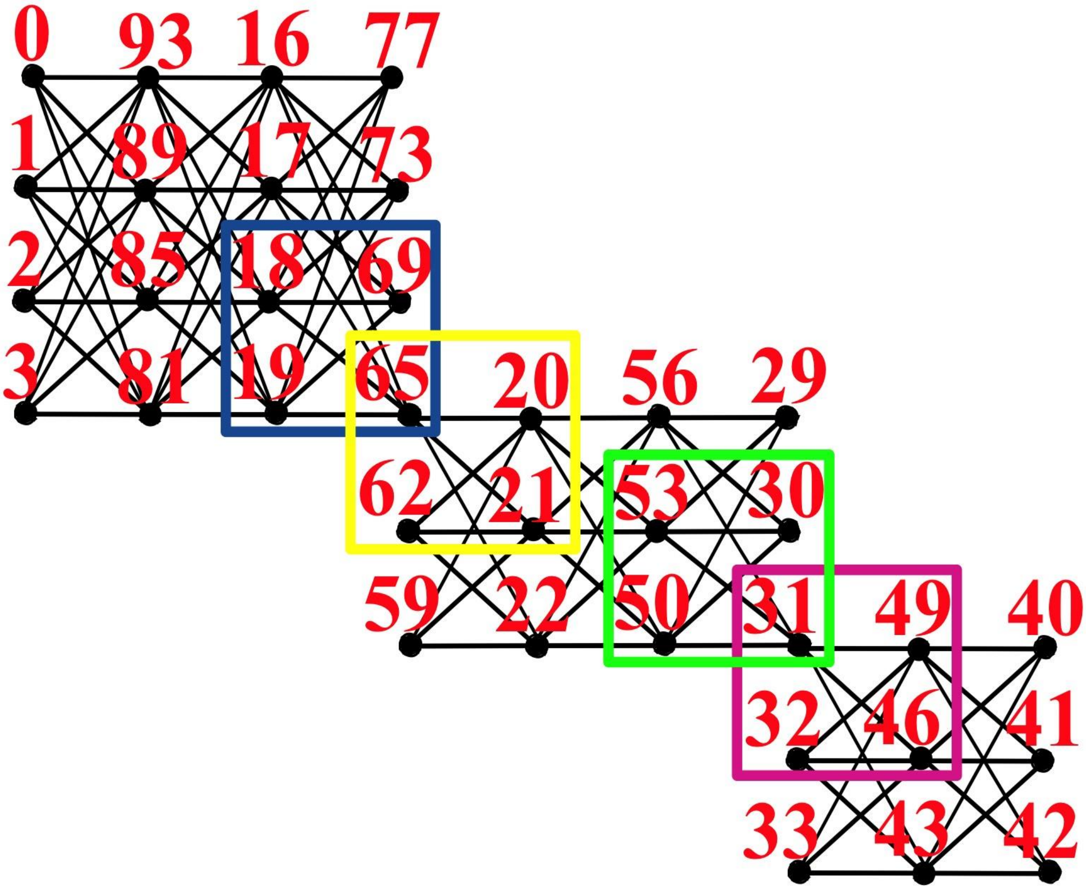

Figure 4. 1 - Graceful labeling of the vertex union of $D_n(P_m)$

Using Theorem 1, we can deduce the proof of this theorem. In order to show that graph $D_n(P_m)$ admits graceful labeling, we need to prove the same thing mentioned in Theorem 1 for column vise and grid vice too. Since the vertex union of $D_n(P_m)$ is an extended version of $D_n(P_m)$ and each grid in Figure 4 has the same characterization which we discussed in the above proof, we can confirm that no edge label repeats in any grid.

Now, we need to show that no edge label repeats in any two grids. For that, consider the blue, yellow, green, and yellow squares in Figure 4. The pink square has the maximum edge labels in the grid $L = 3$, the green square has the minimum edge labels whereas the yellow square has the maximum edge labels of the grid $L = 2$, and the blue square has the minimum edge labels of the grid $L = 1$. Let's compare the edge labels of each square, $blue > yellow > green > pink$. Since the maximum edge labels of $L = 3$ $<$ minimum edge labels of $L = 2$, this implies that all the edge labels of $L = 3$ $<$ all the edge labels of $L = 2$ and by using the same conditions, we can show that any edge label in $L = 2$ $<$ any edge label in $L = 1$.This implies that edge labels vary like this: $(L = 1) > (L = 2) > (L = 3)$. So, we can conclude that two edge labels never repeat throughout the grid.

Using Eqs. (11)-(13), we can show that the maximum edge label of any particular grid equals to $n_L^2(m_L - 1)$. If we take the sum of the maximum edge labels, we get $\sum_{L=1}^{l} n_L^2(m_L - 1)$. Since $\sum_{L=1}^{l} n_L^2(m_L - 1)$ is the total number of edges, we can conclude that this labeling method's maximum edge label does not exceed the total number of edges.

Since we label edges starting from 1 and up until the total number of the edges of that graph and have considered each and every edge label and confirmed that no edge label repeats and the maximum edge label does not exceed the total number of edges, we can conclude that all the edges of that graph must be labeled from 1 up to the total number of edges without repeating. Theorems 5 and 6 can be proved in the same manner.

4 CONCLUSION

Graceful labeling is one of the most engrossing labelings in the literature of graph theory and applications of graceful labeling are even more magnificent. Though there is a large number of research papers in the field of graph theory, there is no certain technique for labeling different classes of graphs. In our research, we proved that triangular-type grid graphs, $D_n(P_m)\ \forall\ m, n$, and L – vertex union of $D_n(P_m)$, admit k - graceful labeling $\forall\ L, m_L, and\ n_L$. We obtained the proof for the gracefulness of these grid graphs from each category and verified that they are graceful. In future work, we hope to apply the k – graceful labeling of this type of triangular-type grid graphs in cryptography and communication networks.

REFERENCES

Abdel-Aal, M. E. (2013). New Classes of Odd Graceful Graphs. *International Journal on Applications of Graph Theory in Wireless Ad Hoc Networks and Sensor Networks*, *5*(2), 3. https://doi.org/10.5121/jgraphoc.2013.5201

Deshmukh, U. N. (2015). IJESRT INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY APPLICATIONS OF GRACEFUL GRAPHS. *International Journal of Engineering Sciences*, *8*. http://www.ijesrt.com

Gallian, J. A. (2009). A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, *16*, 6.

Vaidya, S. K., & Lekha, B. (2010). New Families of Odd Graceful Graphs. *Int. J. Open Problems Compt. Math*, *3*(5). www.i-csrs.org